Derivation of Wien's law by Dimension analysis

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April 28, 2008

Abstract

A dimensional analysis derivation of Wien's formula on black-body radiation by using Stephan-Boltzmann's T^4 law, which also can be derived by classical electromagnetic theory and classical thermodynamics.

1 Introduction

A lot of text books on quantum physics or statistical physics mention Wien's law, but none of them[1] shows its derivation. I guess it is partly because the Wien's original derivation is rather lengthily and complex and students can not learn much from it. Derivation provided here gives simple and short explanation of the law. Although this is not a proof, this derivation has its own advantage. It suggests a universal constant which is known as Planck's action today. This derivation demonstrates another example that dimension analysis gives right answer. Needless to say, dimension analysis is an important practice.

2 Derivation

What we want is energy spectrum density per unit volume $\rho(\omega, T)$. First of all, we know temperature is average energy of the system and always shows up in the form of kT in laws of physics, where k is Boltzmann constant. We are thinking of black body radiation which

[1] which I have :-)

is related to electromagnetism and is not depend on material of black body itself. Let us use speed of light c as well as ω and kT. Rayleigh-Jeans law is obtained by making energy spectrum density from these three quantities.

$$\rho(\omega, T) = A\frac{kT}{c^3}\omega^2, \tag{1}$$

where A is dimensionless. $\rho(\omega, T)$ has to satisfy Stephan-Boltzmann law:

$$u(T) = \int_0^\infty \rho(\omega, T)d\omega = \sigma T^4 \tag{2}$$

Suppose that this dimensionless quantity is in fact a function of ω and T, we can write such function as $A(\omega T^r)$.[2] At this moment we do not know how to make dimensionless quantity just from ω and T, but we will come back later. Putting Eq. 1 into Eq. 2, we get,

$$u(T) = \frac{kT}{c^3}\int_0^\infty \omega^2 A(\omega T^r)d\omega.$$

We want to take T out of the integral, by introducing $y = \omega T^r$,

$$u(T) = \frac{kT}{c^3}\int_0^\infty \frac{y^2}{T^{2r}}A(y)\frac{1}{T^r}dy = \frac{kT^{(1-3r)}}{c^3}\int_0^\infty y^2 A(y)dy.$$

Thus, since the integral is constant, r has to be -1 to meet Stephan-Boltzmann's T^4 law (2). Finally $\rho(\omega, T)$ can be written as follows:

$$\rho(\omega, T) \propto A(\omega/T)\cdot\frac{kT}{c^3}\omega^2 = \omega^3\cdot\frac{A(\omega/T)}{c^3}kT/\omega$$

This is Wien's law. Now we have to come back to the issue mentioned earlier – how to construct dimensionless quantity from ω and T. This issue is now reduced to how to construct dimensionless quantity from ω/T. We know Boltzmann constant which relates temperature to energy. If there is a constant, say $\hbar$, which relates frequency to energy, $\hbar\omega/kT$ become dimensionless.

Appendix: Stephan-Boltzmann's Law

You can find this derivation in many text books on thermodynamics. Here we provide it just to make this article self-contained. From

[2] In case A's dependency is $\omega^s T^r$, we can always rewrite A as $A((\omega T^{r/s})^s)$, and replace A and r by $A'(x) = A(x^s)$ and $r' = r/s$.

Maxwell's equation, one can show[3] radiation pressure is $u(T)/3$, where $u(T)$ is energy density per unit volume:

$$p = u(T)/3, \quad U(T;V) = V u(T).$$

Using above and thermodynamic equation:

$$U(T;V) = T \cdot S(T;V) + F[T;V]$$

We get,

$$\begin{aligned} u(T) = \left(\frac{\partial U}{\partial V}\right)_T &= T\left(\frac{\partial S}{\partial V}\right)_T + \left(\frac{\partial F}{\partial V}\right)_T, \\ &= T\left(\frac{\partial p}{\partial T}\right)_V - p, \\ &= \frac{T}{3}\frac{du(T)}{dT} - \frac{u(T)}{3}. \end{aligned}$$

Here we used thermodynamic relation $p = -(\partial F/\partial V)_T$ and $(\partial S/\partial V)_T = (\partial p/\partial T)_V$. $u(T)$ satisfies the following equation:

$$T\frac{du(T)}{dT} = 4u(T)$$

Thus,

$$u(T) \propto T^4.$$

[3]See e.g.,