Using Force to Derive Maxwell's Equations

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Abstract

A derivation of Maxwell's equations is presented. We start with the relativistic equation of motion and introduce the force tensor $R_{\mu\nu}$ with which the four-force K_μ can be written as $K_\mu = u_\nu R_{\mu\nu}$, where u_μ is the four-velocity. We show that if such a tensor exists it has to be anti-symmetric. Then, we show that the electrostatic force can be written with such an anti-symmetric tensor and introduce the electromagnetic tensor $F_{\mu\nu}$ such that $K_\mu = q\, u_\nu F_{\mu\nu}$. The Lorentz force is derived from the Lorentz transformation of $F_{\mu\nu}$. The first set of Maxwell's equations is derived from $\partial_\nu F_{\mu\nu}$. From the symmetry of the Lorentz transformation of $F_{\mu\nu}$, we introduce dual tensor $\widetilde{F}_{\mu\nu}$ from which we derive the second set of Maxwell's equations. Although we have to introduce four-vector, the calculation effort required is much less than other approaches, because we separated velocity out from the four-force by introducing the force-tensor.

1 Introduction

There are a few other drivations widely available. Schwartz[1] begins with Poisson's equation and transforms it into a Lorentz covariant form by introducing the vector potential and compensating missing terms leading to the electromagnetic tensor. Purcell[2], Elliott[3] and Haskell[4] use Lorentz transformations of electrostatic equations. Purcell's treatment is the most pedagogical while Elliott is more straight forward. Both use ordinary three dimensional vectors. Haskell follows Elliott's procedure but uses four-vector notation.

In this report, we begin with the relativistic equation of motion and introduce force tensor $R_{\mu\nu}$ with which the four-force K_μ can be written as $K_\mu = u_\nu R_{\mu\nu}$, where u_μ is the four-velocity. We show that if such a tensor exists it must be anti-symmetric. Then, we demonstrate that the electrostatic force can be written with such an anti-symmetric tensor and introduce the electromagnetic tensor $F_{\mu\nu}$ such that $K_\mu = q\, u_\nu F_{\mu\nu}$. The Lorentz force is derived from the Lorentz transformation of $F_{\mu\nu}$. The first set of Maxwell's equations is derived from $\partial_\nu F_{\mu\nu}$. From the symmetry of the Lorentz transformation of $F_{\mu\nu}$, we introduce the dual tensor $\tilde{F}_{\mu\nu}$ from which we derive the second set of Maxwell's equations.

This derivation stands similar position as Schwartz's but without introducing potentials. It makes it more evident that the Lorentz force and the Lorentz transformation of fields originates from the force tensor in the Minkowski space. Although we have to introduce four-vector, the calculation effort required is much less than other approaches, because we separated velocity out from the four-force by introducing the force-tensor.

This report is intended for non-experts. We refrain from using advanced mathematics such as the metric tensor. We follow the notation used in the Feynman's Lectures[5], which is summaried in Appendix A.

2 Derivation

2.1 Reference Frames

We consider two inertial frames S and S' such that S' is moving at velocity $\boldsymbol{V}$ relative to S. We will use γ and β defined below.

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}, \qquad \beta = |\boldsymbol{\beta}|\,, \qquad \boldsymbol{\beta} = \boldsymbol{V}/c. \tag{2.1}$$

We will use these only for relative velocity between inertial frames, not for velocity of the particle to avoid confusion. The Lorentz transformation of a four-vector $(x_t, \boldsymbol{x})$ from S to S' is given below.

$$\boldsymbol{x}' = \boldsymbol{x}_\perp + \gamma\left(\boldsymbol{x}_\parallel - \boldsymbol{\beta}\, x_t\right), \qquad x'_t = \gamma\left(x_t - \boldsymbol{\beta}\cdot\boldsymbol{x}\right), \tag{2.2}$$

where $\boldsymbol{x}_\perp$ and $\boldsymbol{x}_\parallel$ is normal and parallel component of $\boldsymbol{x}$ with respect to $\boldsymbol{\beta}$, namely,

$$\boldsymbol{x} = \boldsymbol{x}_\perp + \boldsymbol{x}_\parallel, \qquad \boldsymbol{x}_\parallel = \frac{\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\boldsymbol{x}\right)}{\beta^2}. \tag{2.3}$$

The inverse transformation is obtained by inverting $\boldsymbol{V}$, thus $\boldsymbol{\beta}$.

$$\boldsymbol{x} = \boldsymbol{x}'_\perp + \gamma\left(\boldsymbol{x}'_\parallel + \boldsymbol{\beta}\, x'_t\right), \qquad x_t = \gamma\left(x'_t + \boldsymbol{\beta}\cdot\boldsymbol{x}'\right). \tag{2.4}$$

2.2 Force Tensor

Let us begin with the relativistic equation of motion of a particle whose mass and coordinate are m and $(ct, \boldsymbol{r})$, respectively.

$$mc^2\,\frac{d\,u_\mu}{ds} = K_\mu,$$

where u_μ is the four-velocity of the particle.

$$\mathsf{u} = u_\mu = \frac{d\,(c\,t,\,\boldsymbol{r})}{ds} = \frac{(c\,dt,\,d\boldsymbol{r})}{c\,dt\sqrt{1-(v/c)^2}} = \frac{1}{\sqrt{1-(v/c)^2}}\left(1,\ \frac{\boldsymbol{v}}{c}\right). \tag{2.5}$$

K_μ is the four-force acting on the particle.

$$\mathsf{K} = K_\mu = \frac{1}{\sqrt{1-(v/c)^2}}\left(\frac{\boldsymbol{v}}{c}\cdot\boldsymbol{f},\ \boldsymbol{f}\right) \tag{2.6}$$

Suppose that we can express four-force K_μ by four-velocity u_μ and a tensor $R_{\mu\nu}$, like below.

$$mc^2\,\frac{d\,u_\mu}{ds} = K_\mu = u_\nu R_{\mu\nu}$$

If such $R_{\mu\nu}$ exits, it must be anti-symmetric. The explanation follows.

Taking inner product with u_μ yields

$$mc^2\,u_\mu\frac{d\,u_\mu}{ds} = u_\mu u_\nu R_{\mu\nu}.$$

The left hand side is zero, because $u_\mu u_\mu = 1$.

$$mc^2\,u_\mu\frac{d\,u_\mu}{ds} = \frac{mc^2}{2}\,\frac{d\,u_\mu u_\mu}{ds} = 0$$

Therefore, $u_\mu u_\nu R_{\mu\nu}$ must be zero which means $R_{\mu\nu}$ has to be anti-symmetric.[1]

We get Eq. (2.6) with $\mathsf{R} = R_{\mu\nu} = (\boldsymbol{f},\,0)$.

$$\begin{aligned}\mathsf{uR} &= (u_t,\,\boldsymbol{u})\,(\boldsymbol{f},\ 0)\,,\\ &= \frac{1}{\sqrt{1-(v/c)^2}}\left(\frac{\boldsymbol{v}}{c}\cdot\boldsymbol{f},\ \boldsymbol{f}\right) = \mathsf{K},\end{aligned}$$

where we used Eqs. (A.2) and (2.5).

[1] See Appendix A.2 for more about anti-symmetric tensors.

In case $\mathsf{R} = (\boldsymbol{p}, \boldsymbol{q})$,

$$\mathsf{K} = \mathsf{uR} = (u_t,\, \boldsymbol{u})\,(\boldsymbol{p}, \boldsymbol{q}) = (\boldsymbol{u}\cdot\boldsymbol{p},\; u_t\,\boldsymbol{p} + \boldsymbol{u}\times\boldsymbol{q})\,, \\ = \frac{1}{\sqrt{1-(v/c)^2}}\left(\frac{\boldsymbol{v}}{c}\cdot\boldsymbol{p},\, \boldsymbol{p} + \frac{\boldsymbol{v}}{c}\times\boldsymbol{q}\right).$$

Therefore,

$$\boldsymbol{f} = \boldsymbol{p} + \frac{\boldsymbol{v}}{c}\times\boldsymbol{q}.$$

In case both $\boldsymbol{p}$ and $\boldsymbol{q}$ does not depend of $\boldsymbol{v}$, the Lorentz transformation of $\mathsf{R} = (\boldsymbol{p}, \boldsymbol{q})$ will be, from Eqs. (A.3) and (A.4),

$$\boldsymbol{p}' \;=\; \boldsymbol{p}_{\parallel} \;+\; \gamma\big(\boldsymbol{p} + \boldsymbol{\beta}\times\boldsymbol{q}\big)_{\perp} \tag{2.7}$$

$$\boldsymbol{q}' \;=\; \boldsymbol{q}_{\parallel} \;+\; \gamma\big(\boldsymbol{q} - \boldsymbol{\beta}\times\boldsymbol{p}\big)_{\perp} \tag{2.8}$$

The inverse transformation of above is obtained by changing the sign of $\boldsymbol{\beta}$.

$$\boldsymbol{p} \;=\; \boldsymbol{p}'_{\parallel} \;+\; \gamma\big(\boldsymbol{p}' - \boldsymbol{\beta}\times\boldsymbol{q}'\big)_{\perp}, \tag{2.9}$$

$$\boldsymbol{q} \;=\; \boldsymbol{q}'_{\parallel} \;+\; \gamma\big(\boldsymbol{q}' + \boldsymbol{\beta}\times\boldsymbol{p}'\big)_{\perp}. \tag{2.10}$$

2.3 Maxwell's Equations

We assume continuity of charge holds in any inertial frame,

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{j} = 0,$$

where ρ and $\boldsymbol{j}$ are charge density and current density, respectively. This leads to ρ and $\boldsymbol{j}$ forming a four-vector $\mathsf{j} = (c\rho,\, \boldsymbol{j})$, because this continuity equation can be viewed as the inner product of the four-gradient and the four-current is zero, or a scalar.

Suppose that all the charge is at rest in S', i.e., $\boldsymbol{j}' = 0$.

$$\mathsf{j}' = \big(c\rho',\, \boldsymbol{j}'\big)\,, \quad \boldsymbol{j}' = 0. \tag{2.11}$$

Static charge density ρ' creates static electric field $\boldsymbol{E}'$.

$$\boldsymbol{\nabla}'\cdot\boldsymbol{E}' = \rho'/\varepsilon_0, \quad \boldsymbol{\nabla}'\times\boldsymbol{E}' = 0 \tag{2.12}$$

Let us consider motion of a charged particle whose charge q is very small so that its motion will not affect the rest of charges and thus the charge density. We assume q is Lorentz invariant, i.e., $q' = q$. The force acting on the charged particle $\boldsymbol{f}'$ is

$$\boldsymbol{f}' = q\boldsymbol{E}'.$$

Since $\boldsymbol{E}'$ does not depend on the velocity $\boldsymbol{v}$ of the particle, we can use the force tensor.

$$\mathsf{K}' = q\,(u'_t, \boldsymbol{u}')(\boldsymbol{E}',\ 0).$$

In S, all charges except for q are moving uniformly at $\boldsymbol{V}$, four-force K is, according to Eqs. (2.9) and (2.10),

$$\mathsf{K} = q\,(u_t, \boldsymbol{u})(\boldsymbol{E}'_{\parallel}\ +\ \gamma \boldsymbol{E}'_{\perp},\ \boldsymbol{\beta} \times \gamma \boldsymbol{E}').$$

This gives

$$\boldsymbol{f}\ =\ q\Big(\boldsymbol{E}'_{\parallel}\ +\ \gamma \boldsymbol{E}'_{\perp}\ +\ \frac{\boldsymbol{v}}{c} \times \big(\boldsymbol{\beta} \times \gamma \boldsymbol{E}'\big)\Big).$$

$(\boldsymbol{E}'_{\parallel} + \gamma \boldsymbol{E}'_{\perp})$ is electric field in S and $(\boldsymbol{\beta} \times \gamma \boldsymbol{E}')$ can be interpreted as magnetic field.

Now, we define the electromagnetic tensor F as below.[2]

$$c\mathsf{F} = (\boldsymbol{E},\ c\boldsymbol{B})$$

Giving four-velocity of the particle u to F yields four-force K.

$$\begin{aligned} q\,\mathsf{u}\,c\mathsf{F} = \mathsf{K} = q\,(u_t, \boldsymbol{u})\,(\boldsymbol{E}, c\boldsymbol{B}) &= q\,(\boldsymbol{u} \cdot \boldsymbol{E},\ u_t \boldsymbol{E} + \boldsymbol{u} \times c\boldsymbol{B})\,, \\ &= \frac{q}{\sqrt{1-(v/c)^2}} \left(\frac{\boldsymbol{v}}{c} \cdot \boldsymbol{E},\ \boldsymbol{E} + \frac{\boldsymbol{v}}{c} \times c\boldsymbol{B} \right) \end{aligned}$$

Applying the four-gradient $\mathsf{\partial}$ to F yields a four-vector.

$$\mathsf{\partial}\, c\mathsf{F} = \left(\frac{1}{c}\frac{\partial}{\partial t},\ -\boldsymbol{\nabla} \right) (\boldsymbol{E}, c\boldsymbol{B}) = \left(-\boldsymbol{\nabla} \cdot \boldsymbol{E},\ \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} - \boldsymbol{\nabla} \times c\boldsymbol{B} \right)$$

Recalling that $\boldsymbol{\nabla}' \cdot \boldsymbol{E}' = \rho'/\varepsilon_0$ in S', we see that the time component of the above four-vector is equal to $-\rho/\varepsilon_0$. Then, the space component has to be equal to $-\boldsymbol{j}/\varepsilon_0 c$ to make it a four-vector. Therefore,

$$\mathsf{\partial}\, c\mathsf{F} = -\mathsf{j}/\varepsilon_0 c, \quad \text{or} \quad \partial_\nu\, cF_{\mu\nu} = -j_\mu/\varepsilon_0 c,$$

or in ordinary vector,

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0}, \qquad \boldsymbol{\nabla} \times c\boldsymbol{B} - \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}\ =\ \frac{\boldsymbol{j}}{\varepsilon_0 c}. \tag{2.13}$$

$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho/\epsilon_0$ holds not only for static field but also for $\boldsymbol{E}$ created by moving charge.

[2] We follow SI units. It will be $\boldsymbol{B}$ instead of $c\boldsymbol{B}$ in Gaussian units. See Ref. [6] for details.

The Lorentz transformation of $\boldsymbol{E}$ and $c\boldsymbol{B}$ is, according to Eqs. (2.7) and (2.8),

$$\boldsymbol{E}' = \boldsymbol{E}_{\parallel} + \gamma\big(\boldsymbol{E} + \boldsymbol{\beta}\times c\boldsymbol{B}\big)_{\perp}, \tag{2.14}$$

$$c\boldsymbol{B}' = c\boldsymbol{B}_{\parallel} + \gamma\big(c\boldsymbol{B} - \boldsymbol{\beta}\times \boldsymbol{E}\big)_{\perp}. \tag{2.15}$$

We see symmetry between $\boldsymbol{E}$ and $c\boldsymbol{B}$ in above Lorentz transformation. The Lorentz transformation remains the same under following substitution.

$$\boldsymbol{E} \rightarrow c\boldsymbol{B}, \qquad c\boldsymbol{B} \rightarrow -\boldsymbol{E},$$

Therefore, we have another anti-symmetric tensor, which gives the same Lorentz transformation for $\boldsymbol{E}$ and $c\boldsymbol{B}$. That is

$$c\tilde{\mathsf{F}} = (c\boldsymbol{B}, -\boldsymbol{E})\,.$$

Applying four-gradient $\boldsymbol{\partial}$ to $\tilde{\mathsf{F}}$ yields a four-vector.

$$\boldsymbol{\partial}\, c\tilde{\mathsf{F}} = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\boldsymbol{\nabla}\right)(c\boldsymbol{B}, -\boldsymbol{E}) = \left(-\boldsymbol{\nabla}\cdot c\boldsymbol{B},\ \frac{1}{c}\frac{\partial c\boldsymbol{B}}{\partial t} + \boldsymbol{\nabla}\times\boldsymbol{E}\right)$$

In S', we don't have any ordinary force component that depends on the velocity of the particle, therefore $c\boldsymbol{B}' = 0$. $\boldsymbol{E}'$ is static, $\boldsymbol{\nabla}'\times\boldsymbol{E}' = 0$. Therefore, this vector is zero in S'. Because the Lorentz transformation is a linear transformation, this vector must be zero in any inertial frame. Therefore

$$\boldsymbol{\partial}\, c\tilde{\mathsf{F}} = 0, \quad \text{or} \quad \partial_\nu\, c\tilde{F}_{\mu\nu} = 0,$$

or in ordinary vector,

$$\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0, \qquad \boldsymbol{\nabla}\times\boldsymbol{E} + \frac{1}{c}\frac{\partial c\boldsymbol{B}}{\partial t} = 0. \tag{2.16}$$

Eqs. (2.13) and (2.16) are the Maxwell's equations.

3 Concluding Remarks

We assumed following physical properties of electricity.

(a) Charge of the particle is Lorentz invariant. That is $q' = q$.

(b) Continuity of charge holds in any inertial frame.

(c) $\boldsymbol{E}'$ is static. $\boldsymbol{\nabla}'\cdot\boldsymbol{E}' = \rho'/\epsilon_0$, $\boldsymbol{\nabla}'\times\boldsymbol{E}' = 0$, and $\mathsf{j}' = (c\rho', 0)$.

(d) Force $\boldsymbol{f}'$ acting on moving charged particle in static electric field $\boldsymbol{E}'$ is $\boldsymbol{f}' = q\boldsymbol{E}'$.

(b) can be derived from (a) if we employ following model.

$$\rho = \sum_i q_i \, \delta^3(\boldsymbol{r} - \boldsymbol{r}_i), \qquad \boldsymbol{j} = \sum_i \boldsymbol{v}_i \, q_i \, \delta^3(\boldsymbol{r} - \boldsymbol{r}_i).$$

(c) is consequence of Coulomb's law. (d) as well as Maxwell's equations derived here are also subject to verification. Jackson[7] has good review of validity of the theory. Feynman points out some of fundamental challenges of the theory in his Lectures[5].

Appendix A Vectors and tensors

This report is intended for non-experts. We refrain from using advanced mathematics such as metric tensor. We follow the notation used in the Feynman's Lectures[5].

A.1 Vectors

We use following notation for ordinary vectors. We use i, j, k, ... for index x, y, z.

$$\boldsymbol{a} = a_i = \begin{pmatrix} a_x & a_y & a_z \end{pmatrix}$$

Below is inner product of ordinary vectors. Summation rule applies.

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_i b_i = a_x b_x + a_y b_y + a_z b_z$$

We use following notation for four-vector. We use μ, ν, ρ, ... for index t, x, y, z.

$$\mathsf{a} = (a_t, \boldsymbol{a}) = a_\mu = \begin{pmatrix} a_t & a_x & a_y & a_z \end{pmatrix}$$

Below is inner product of four-vectors. Summation rule applies.

$$\begin{aligned} \mathsf{a} \cdot \mathsf{b} &= (a_t, \boldsymbol{a}) \cdot (b_t, \boldsymbol{b}), \\ &= a_\mu b_\mu, \\ &= a_t b_t - \boldsymbol{a} \cdot \boldsymbol{b}, \\ &= a_t b_t - a_x b_x - a_y b_y - a_z b_z \end{aligned}$$

A.2 Anti-symmetric tensor

An anti-symmetric tensor is a second-rank tensor whose component flips its sign when its index order is flipped, like below.

$$A_{\mu\nu} = -A_{\mu\nu}$$

It has six independent components.

$$A_{\mu\nu} = \begin{pmatrix} 0 & A_{tx} & A_{ty} & A_{tz} \\ A_{xt} & 0 & A_{xy} & A_{xz} \\ A_{yt} & A_{yx} & 0 & A_{yz} \\ A_{zt} & A_{zx} & A_{zy} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -A_{xt} & -A_{yt} & -A_{zt} \\ A_{xt} & 0 & -A_{yx} & A_{xz} \\ A_{yt} & A_{yx} & 0 & -A_{zy} \\ A_{zt} & -A_{xz} & A_{zy} & 0 \end{pmatrix}$$

Let us work on the case when it is written with two four-vectors as below.

$$A_{\mu\nu} = a_\mu b_\nu - a_\nu b_\mu$$

Because any anti-symmetric tensor can be written with a linear combination of these, extending below discussion to general case is straight forward.

A.2.1 Two ordinary vectors

$\boldsymbol{p} = (A_{xt}\ A_{yt}\ A_{zt})$ forms an ordinary vector because it is linear combination of two ordinary vectors (or linear combination of those).

$$\begin{aligned} \boldsymbol{p} &= \begin{pmatrix} A_{xt} & A_{yt} & A_{zt} \end{pmatrix} \\ &= \begin{pmatrix} a_x b_t - a_t b_x & a_y b_t - a_t b_y & a_z b_t - a_t b_z \end{pmatrix}, \\ &= \begin{pmatrix} a_x & a_y & a_z \end{pmatrix} b_t - a_t \begin{pmatrix} b_x & b_y & b_z \end{pmatrix}, \\ &= \boldsymbol{a}\, b_t - a_t\, \boldsymbol{b} \end{aligned}$$

$\boldsymbol{q} = (A_{zy}\ A_{xz}\ A_{yx})$ forms an ordinary vector too, because it is vector product of two ordinary vectors (or linear combination of those).

$$\begin{aligned} \boldsymbol{q} &= \begin{pmatrix} A_{zy} & A_{xz} & A_{yx} \end{pmatrix}, \\ &= \begin{pmatrix} (a_z b_y - a_y b_z) & (a_x b_z - a_z b_x) & (a_y b_x - a_x b_y) \end{pmatrix}, \\ &= -\boldsymbol{a} \times \boldsymbol{b} \end{aligned}$$

We use $(\boldsymbol{p}, \boldsymbol{q})$ to represent A.

$$\mathsf{A} = (\boldsymbol{p}, \boldsymbol{q}) = A_{\mu\nu} = \begin{pmatrix} 0 & -p_x & -p_y & -p_z \\ p_x & 0 & -q_z & q_y \\ p_y & q_z & 0 & -q_x \\ p_z & -q_y & q_x & 0 \end{pmatrix} \tag{A.1}$$

When both $\boldsymbol{a}$ and $\boldsymbol{b}$ are polar, $\boldsymbol{p}$ is polar and $\boldsymbol{q}$ is axial.

A.2.2 Linear map

Giving four-vector w to anti-symmetric tensor A yields a four-vector.

$$\begin{aligned}\mathsf{wA} = (w_t, \boldsymbol{w})\,(\boldsymbol{p}, \boldsymbol{q}) &= w_\nu A_{\mu\nu} \\ &= w_t A_{\mu t} - w_x A_{\mu x} - w_y A_{\mu y} - w_z A_{\mu z}.\end{aligned}$$

t, x, y, z components are, respectively,

$$\begin{aligned}w_\nu A_{t\nu} &= w_t A_{tt} - w_x A_{tx} - w_y A_{ty} - w_z A_{tz}, \\ &= w_x\, p_x + w_y\, p_y + w_z\, p_z = \boldsymbol{w}\cdot\boldsymbol{p}, \\ w_\nu A_{x\nu} &= w_t A_{xt} - w_x A_{xx} - w_y A_{xy} - w_z A_{xz}, \\ &= w_t\, p_x + w_y\, q_z - w_z\, q_y, \\ w_\nu A_{y\nu} &= w_t A_{yt} - w_x A_{yx} - w_y A_{yy} - w_z A_{yz}, \\ &= w_t\, p_y + w_z\, q_x - w_x\, q_z, \\ w_\nu A_{z\nu} &= w_t A_{zt} - w_x A_{zx} - w_y A_{zy} - w_z A_{zz}, \\ &= w_t\, p_z + w_x\, q_y - w_y\, q_x.\end{aligned}$$

Therefore

$$\mathsf{wA} = (w_t, \boldsymbol{w})\,(\boldsymbol{p}, \boldsymbol{q}) = w_\nu A_{\mu\nu} = (\boldsymbol{w}\cdot\boldsymbol{p},\ w_t\,\boldsymbol{p} + \boldsymbol{w}\times\boldsymbol{q})\,. \tag{A.2}$$

A.2.3 Lorentz transformation

Let us use Cartesian coordinate whose x-axis coincides $\boldsymbol{\beta}$, i.e., $\boldsymbol{\beta} = (\beta_x\ 0\ 0)$. The Lorentz transformation of $\boldsymbol{a}$ and $\boldsymbol{b}$ is, respectively,

$$a'_x = \gamma\,(a_x - \beta_x\, a_t)\,, \quad a'_y = a_y, \quad a'_z = a_z, \quad a'_t = \gamma\,(a_t - \beta_x\, a_x)\,,$$

and

$$b'_x = \gamma\,(b_x - \beta_x\, b_t)\,, \quad b'_y = b_y, \quad b'_z = b_z, \quad b'_t = \gamma\,(b_t - \beta_x\, b_x)\,.$$

For $\boldsymbol{p} = (A_{xt}\ A_{yt}\ A_{zt})$,

$$\begin{aligned}p'_x = A'_{xt} &= a'_x b'_t - a'_t b'_x, \\ &= \gamma^2\,(a_x - \beta_x\, a_t)\,(b_t - \beta_x\, b_x) - \gamma^2\,(a_t - \beta_x\, a_x)\,(b_x - \beta_x\, b_t)\,, \\ &= \gamma^2 (1-\beta_x^2)\, a_x b_t - \gamma^2 (1-\beta_x^2)\, a_t b_x = a_x b_t - a_t b_x \\ &= A_{xt} = p_x, \\ p'_y = A'_{yt} &= a'_y b'_t - a'_t b'_y, \\ &= \gamma\, a_y\,(b_t - \beta_x\, b_x) - \gamma\ (a_t - \beta_x\, a_x)\, b_y, \\ &= \gamma\,(a_y b_t - a_t b_y - \beta_x\, a_y b_x + \beta_x\, a_x b_y) \\ &= \gamma\,(A_{yt} - \beta_x\, A_{yx}) = \gamma\,(p_y - \beta_x\, q_z)\,, \\ p'_z = A'_{zt} &= \gamma\,(A_{zt} + \beta_x\, A_{xz}) = \gamma\,(p_z + \beta_x\, q_y)\,.\end{aligned}$$

Similarly, for $\boldsymbol{q} = (A_{zy}\ A_{xz}\ A_{yx})$,

$$\begin{aligned} q'_x &= A'_{zy} = A_{zy} = q_x, \\ q'_y &= A'_{xz} = \gamma\left(A_{xz} + \beta_x\, A_{zt}\right) = \gamma\left(q_y + \beta_x\, p_z\right), \\ q'_z &= A'_{yx} = \gamma\left(A_{yx} - \beta_x\, A_{yt}\right) = \gamma\left(q_z - \beta_x\, p_y\right). \end{aligned}$$

To convert these to the vector notation, we need to recover missing terms. For example, when z component has term $\beta_x\, q_y$, we need to add $-\beta_y\, q_x$ to make it a vector. Then,

$$\boldsymbol{p}'_{\parallel} = \boldsymbol{p}_{\parallel}, \quad \boldsymbol{p}'_{\perp} = \gamma\left(\boldsymbol{p} + \boldsymbol{\beta} \times \boldsymbol{q}\right)_{\perp}, \tag{A.3}$$

$$\boldsymbol{q}'_{\parallel} = \boldsymbol{q}_{\parallel}, \quad \boldsymbol{q}'_{\perp} = \gamma\left(\boldsymbol{q} - \boldsymbol{\beta} \times \boldsymbol{p}\right)_{\perp}. \tag{A.4}$$

The inverse transformation can be obtained by inverting the sign of $\boldsymbol{\beta}$.

$$\boldsymbol{p}_{\parallel} = \boldsymbol{p}'_{\parallel}, \quad \boldsymbol{p}_{\perp} = \gamma\left(\boldsymbol{p}' - \boldsymbol{\beta} \times \boldsymbol{q}'\right)_{\perp}, \tag{A.5}$$

$$\boldsymbol{q}_{\parallel} = \boldsymbol{q}'_{\parallel}, \quad \boldsymbol{q}_{\perp} = \gamma\left(\boldsymbol{q}' + \boldsymbol{\beta} \times \boldsymbol{p}'\right)_{\perp}. \tag{A.6}$$

A.2.4 Dual tensor

We see symmetry between $\boldsymbol{p}$ and $\boldsymbol{q}$ in their Lorentz transformation (Eqs. (A.3), (A.4)), i.e., following substitution will give the same Lorentz transformation.

$$\boldsymbol{p} \ \rightarrow\ \boldsymbol{q}, \quad \boldsymbol{q} \ \rightarrow\ -\boldsymbol{p}.$$

Therefore, $\tilde{\mathsf{A}} = (\boldsymbol{q}, -\boldsymbol{p})$ gives the same transformation for $\boldsymbol{p}$ and $\boldsymbol{q}$ as $\mathsf{A} = (\boldsymbol{p}, \boldsymbol{q})$.

$$\tilde{\mathsf{A}} = (\boldsymbol{q}, -\boldsymbol{p}) = \tilde{A}_{\mu\nu} = \begin{pmatrix} 0 & -q_x & -q_y & -q_z \\ q_x & 0 & p_z & -p_y \\ q_y & -p_z & 0 & p_x \\ q_z & p_y & -p_x & 0 \end{pmatrix} \tag{A.7}$$

Dual tensor is valid only under "proper" (not inverting) coordinate transformations, since we are substituting polar vector with axial vector, vice versa.

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