Five Derivations of Lorentz Transformation

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Abstract

Five derivations of the Lorentz transformation are presented: 1) wave equation, 2) constancy of the speed of light, 3) rotation analogy, 4) symmetry and transformation group, 5) vector notation. In the first four derivations, we consider only linear transformations of time t and a one dimensional spatial coordinate x. Justification will be provided in the forth derivation. For the final derivation, we will use three dimensional vectors for spatial coordinates.

1 Five Derivations of Lorentz Transformation

In this report, we will derive the Lorentz transformation in 5 different ways, 1) wave equation, 2) constancy of the speed of light, 3) rotation analogy, 4) symmetry and transformation group, 5) vector notation. In the first four derivations, we consider only linear transformations of time t and a one dimensional spatial coordinate x. Justification for this will be provided in the forth derivation, Section 1.4. For the final derivation, we will use three dimensional vectors for spatial coordinates.

Let's consider an inertial frame K with coordinates (ct, x), and another inertial frame K' with coordinates (ct', x'). Suppose that K' is moving at velocity V relative to K and the origin of K' coincides with the origin of K at $t = 0$. A trivial condition is that a point $x = Vt$ is transformed to $x' = 0$. Therefore, the transformation for x can be written as

$$x' = \alpha \left(x - (V/c)\, ct\right).$$

Here, we will treat ct as a single symbol to ensure it has the same dimension as x (length). Now, we have only one parameter α for the spatial coordinate. For ct, we introduce two parameters β and γ, as follows:

$$ct' = \gamma\,(ct - \beta\,x)$$

Our task is to find three parameters, α, β and γ, under the given conditions.

1.1 Wave equation

We require that the condition that the wave equation

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} = 0.$$

is invariant under a coordinate transformation

$$x' = \alpha\,(x - (V/c)\,ct)\,, \qquad ct' = \gamma\,(ct - \beta\,x)\,, \tag{1.1}$$

where V is the velocity of K' relative to K. The first term of the wave equation transforms as

$$\begin{aligned}\frac{\partial^2}{\partial ct^2} &= \frac{\partial}{\partial ct}\left(\frac{\partial ct'}{\partial ct}\frac{\partial}{\partial ct'} + \frac{\partial x'}{\partial ct}\frac{\partial}{\partial x'}\right), \\ &= \left(\frac{\partial ct'}{\partial ct}\right)^2\frac{\partial^2}{\partial ct'^2} + \left(\frac{\partial x'}{\partial ct}\right)^2\frac{\partial^2}{\partial x'^2} + 2\left(\frac{\partial ct'}{\partial ct}\right)\left(\frac{\partial x'}{\partial ct}\right)\frac{\partial^2}{\partial ct'\,\partial x'} \end{aligned}\tag{1.2}$$

The second term of the wave equation transforms as

$$\begin{aligned}\frac{\partial^2}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial ct'}{\partial x}\frac{\partial}{\partial ct'} + \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'}\right), \\ &= \left(\frac{\partial ct'}{\partial x}\right)^2\frac{\partial^2}{\partial ct'^2} + \left(\frac{\partial x'}{\partial x}\right)^2\frac{\partial^2}{\partial x'^2} + 2\left(\frac{\partial ct'}{\partial x}\right)\left(\frac{\partial x'}{\partial x}\right)\frac{\partial^2}{\partial ct'\,\partial x'} \end{aligned}\tag{1.3}$$

Here from Eqs. (1.1),

$$\left(\frac{\partial ct'}{\partial ct}\right) = \gamma, \quad \left(\frac{\partial x'}{\partial ct}\right) = -\alpha\,V/c, \quad \left(\frac{\partial ct'}{\partial x}\right) = -\gamma\beta, \quad \left(\frac{\partial x'}{\partial x}\right) = \alpha.$$

For the wave equation to be Lorentz invariant, we require Eq. (1.2) minus Eq. (1.3) becomes

$$\frac{\partial^2}{\partial ct^2} - \frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial ct'^2} - \frac{\partial^2}{\partial x'^2}.$$

In other words, we require the coefficient of $\partial^2/\partial ct'^2$, $\partial^2/\partial x'^2$ and $\partial^2/\partial ct'\partial x'$ to be 1, -1 and 0, respectively. That is

$$\gamma^2 - \gamma^2\beta^2 = 1, \quad \alpha^2 \left(V/c\right)^2 - \alpha^2 = -1, \quad -\gamma\alpha V/c + \gamma\beta\alpha = 0.$$

The solution is

$$\beta = V/c, \quad \gamma^2 = \alpha^2 = \frac{1}{1-\beta^2}.$$

Because we want the transformation to be the identity when $V = 0$, we take positive values for both α and γ.

$$\gamma = \alpha = \frac{1}{\sqrt{1-(V/c)^2}}$$

1.2 Constancy of the speed of light

We begin with transformation of (cdt, dx).

$$dx' = \alpha \left(dx - (V/c)\, cdt\right), \quad cdt' = \gamma \left(cdt - \beta\, dx\right).$$

We want to find α, β, γ which keeps the interval $(cdt)^2 - (dx)^2$ constant. They are calculated as follows:

$$\begin{aligned}(cdt')^2 &= \gamma^2 \left((cdt)^2 + \beta^2 (dx)^2 - 2\,\beta\, cdt\, dx\right),\\ (dx')^2 &= \alpha^2 \left((dx)^2 + (V/c)^2 (cdt)^2 - 2\,(V/c)\, cdt\, dx\right).\end{aligned}$$

The interval in K' is

$$\begin{aligned}(cdt')^2 - (dx')^2 = \left(\gamma^2 - \alpha^2 (V/c)^2\right)(cdt)^2 - \left(\alpha^2 - \gamma^2\beta^2\right)(dx)^2 \\ + 2\left(-\gamma^2\beta + \alpha^2\, V/c\right) cdt\, dx.\end{aligned}$$

This must be equal to the interval in K, $(cdt)^2 - (dx)^2$. Therefore

$$\gamma^2 - \alpha^2 (V/c)^2 = 1, \quad \alpha^2 - \gamma^2\beta^2 = 1, \quad -\gamma^2\beta + \alpha^2\, V/c = 0.$$

The solution is

$$\beta = V/c, \quad \gamma^2 = \alpha^2 = \frac{1}{1-\beta^2}.$$

Because we want the transformation to be the identity when $V = 0$, we take positive values for both α and γ.

$$\gamma = \alpha = \frac{1}{\sqrt{1-(V/c)^2}}$$

1.3 Rotation analogy

In a 2-dimensional vector space, it is rotation that keeps $x^2 + y^2$ constant.

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Here, we want to find a transformation that keeps $(ct)^2 - x^2$ constant. Let's consider the following transformation.

$$\begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} ct' \\ x' \end{pmatrix} \tag{1.4}$$

One can easily verify that this transformation indeed gives $(ct)^2 - x^2 = (ct')^2 - x'^2$.

Let (ct, x) be the coordinate of the origin of K' in K. Since the coordinate of the origin of K' in K' is $(ct', 0)$,

$$\begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} ct' \\ 0 \end{pmatrix}.$$

Therefore

$$ct = ct' \cosh\theta, \quad x = ct' \sinh\theta$$

Since O' is moving at V in K, i.e., $V = x/t$,

$$\frac{V}{c} = \frac{x}{ct} = \frac{\sinh\theta}{\cosh\theta} = \tanh\theta.$$

Recalling that $\cosh^2\theta - \sinh^2\theta = 1$, we find

$$\cosh\theta = \frac{1}{\sqrt{1 - \tanh^2\theta}}, \quad \sinh\theta = \frac{\tanh\theta}{\sqrt{1 - \tanh^2\theta}}.$$

Therefore

$$\cosh\theta = \frac{1}{\sqrt{1 - (V/c)^2}}, \quad \sinh\theta = \frac{V/c}{\sqrt{1 - (V/c)^2}}.$$

Inserting these to Eq. (1.4),

$$\begin{pmatrix} ct \\ x \end{pmatrix} = \frac{1}{\sqrt{1 - (V/c)^2}} \begin{pmatrix} 1 & V/c \\ V/c & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \end{pmatrix}.$$

The inverse transformation will be

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \frac{1}{\sqrt{1 - (V/c)^2}} \begin{pmatrix} 1 & -V/c \\ -V/c & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}.$$

1.4 Symmetry and transformation group

1.4.1 Introduction

In this section, we derive the Lorentz transformation by considering the symmetry of space-time and transformation group. It was constancy of the speed of light which brought the idea that time must be transformed along with the spatial coordinates. Once we accept this idea, we can demonstrate that the transformation from one inertial frame to another is either a Lorentz transformation or a Galilean transformation, based on a few assumption of space, time and the transformation itself. Those are 1) isotropy of space, 2) homogeneity of space and time, 3) the transformation forms a transformation group. [1, 2]

The outline of the derivation is as follows: We first show that the transformation is linear and separates into (t, x) and (y, z) coordinates from the isotropy of space and the homogeneity of space and time, both of which were implicitly assumed in the previous derivations in this report. Then, the reciprocity of the transformation and combined transformation are applied to determine the matrix element of the transformation. The resulting transformation includes an arbitrary parameter K, which is identified as $1/c^2$ for the transformation to be the Lorentz transformation.

1.4.2 Derivation

Consider a Cartesian coordinate system S' moving at a velocity v relative to another Cartesian coordinate system S. Suppose that the x, y, z axes of S and the x', y', z' axes of S' coincides at $t = 0$ and the transformation of spatial and time coordinate (t, x, y, z) of S to spatial and time coordinate (t', x', y', z') of S' is written as follows:

$$
\begin{aligned}
t' &= T(t, x, y, z; v), \\
x' &= X(t, x, y, z; v), \\
y' &= Y(t, x, y, z; v), \\
z' &= Z(t, x, y, z; v).
\end{aligned}
$$

The transformation should be the identity when $v = 0$.

$$
\begin{aligned}
T(t, x, y, z; 0) &= t, \\
X(t, x, y, z; 0) &= x, \\
Y(t, x, y, z; 0) &= y, \\
Z(t, x, y, z; 0) &= z.
\end{aligned} \tag{1.5}
$$

From the isotropy of space, the directions of axes are arbitrary as long as they are orthogonal each other. However, since we set the direction of relative velocity v to x-axis, we only consider rotations that will not alter the direction of x-axis. If we rotate the y and z axes of S around the x-axis by θ, the coordinates of a point (x, y, z) will become $(x, y\cos\theta + z\sin\theta, -y\sin\theta + z\cos\theta)$. The same thing can be said for S'. Therefore

$$\begin{aligned}
T(t, x, y\cos\theta + z\sin\theta, -y\sin\theta + z\cos\theta; v) &= t', \\
X(t, x, y\cos\theta + z\sin\theta, -y\sin\theta + z\cos\theta; v) &= x', \\
Y(t, x, y\cos\theta + z\sin\theta, -y\sin\theta + z\cos\theta; v) &= y'\cos\theta + z'\sin\theta, \\
Z(t, x, y\cos\theta + z\sin\theta, -y\sin\theta + z\cos\theta; v) &= -y'\sin\theta + z'\cos\theta.
\end{aligned}$$

In case $\theta = 180°$,

$$\begin{aligned}
T(t, x, -y, -z; v) &= t', \\
X(t, x, -y, -z; v) &= x', \\
Y(t, x, -y, -z; v) &= -y', \\
Z(t, x, -y, -z; v) &= -z'.
\end{aligned} \tag{1.6}$$

Now, let's consider a rotation around the z-axis. This time, we can only perform a $180°$ rotation since we don't want to alter the direction of the x-axis. If we rotate the x and y axes of S around the z-axis by $180°$, the coordinates of a point (x, y, z) will become $(-x, -y, z)$. The same can be said for S'. This time, the x-axis got inverted, the velocity v must also be inverted.

$$\begin{aligned}
T(t, -x, -y, z; -v) &= t', \\
X(t, -x, -y, z; -v) &= -x', \\
Y(t, -x, -y, z; -v) &= -y', \\
Z(t, -x, -y, z; -v) &= z'.
\end{aligned} \tag{1.7}$$

From the homogeneity of space and time, all functions have translation symmetry. For example, for two time points t_2 and t_1,

$$T(x, y, z, t_2) - T(x, y, z, t_1) = T(x, y, z, t_2 + t) - T(x, y, z, t_1 + t),$$

for any t. Rearranging above yields

$$T(x, y, z, t_1 + t) - T(x, y, z, t_1) = T(x, y, z, t_2 + t) - T(x, y, z, t_2).$$

Since this is true for any t_1 and t_2, T must be a linear function of t.

$$T(t) = a\,t + const.$$

The same applies to all coordinates (t, x, y, z) and functions (T, X, Y, Z). Therefore, the transformation can be written in a matrix form as follows:

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} a_{tt}(v) & a_{tx}(v) & a_{ty}(v) & a_{tz}(v) \\ a_{xt}(v) & a_{xx}(v) & a_{xy}(v) & a_{xz}(v) \\ a_{yt}(v) & a_{yx}(v) & a_{yy}(v) & a_{yz}(v) \\ a_{zt}(v) & a_{zx}(v) & a_{zy}(v) & a_{zz}(v) \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

From Eq. (1.6), we have

$$\begin{pmatrix} t' \\ x' \\ -y' \\ -z' \end{pmatrix} = \begin{pmatrix} a_{tt}(v) & a_{tx}(v) & a_{ty}(v) & a_{tz}(v) \\ a_{xt}(v) & a_{xx}(v) & a_{xy}(v) & a_{xz}(v) \\ a_{yt}(v) & a_{yx}(v) & a_{yy}(v) & a_{yz}(v) \\ a_{zt}(v) & a_{zx}(v) & a_{zy}(v) & a_{zz}(v) \end{pmatrix} \begin{pmatrix} t \\ x \\ -y \\ -z \end{pmatrix}$$

By comparing the above two expressions, we see some of coefficients must be zero.

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} a_{tt}(v) & a_{tx}(v) & 0 & 0 \\ a_{xt}(v) & a_{xx}(v) & 0 & 0 \\ 0 & 0 & a_{yy}(v) & a_{yz}(v) \\ 0 & 0 & a_{zy}(v) & a_{zz}(v) \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

From Eq. (1.7), we have

$$\begin{pmatrix} t' \\ -x' \\ -y' \\ z' \end{pmatrix} = \begin{pmatrix} a_{tt}(-v) & a_{tx}(-v) & 0 & 0 \\ a_{xt}(-v) & a_{xx}(-v) & 0 & 0 \\ 0 & 0 & a_{yy}(-v) & a_{yz}(-v) \\ 0 & 0 & a_{zy}(-v) & a_{zz}(-v) \end{pmatrix} \begin{pmatrix} t \\ -x \\ -y \\ z \end{pmatrix}$$

By comparing the above two expressions, we see that

$$\begin{gathered} a_{yy}(v) = a_{zz}(v) = a_{yy}(-v) = a_{zz}(-v), \\ a_{yz}(v) = a_{zy}(v) = a_{yz}(-v) = a_{zy}(-v) = 0, \\ a_{tt}(-v) = a_{tt}(v), \quad a_{tx}(-v) = -a_{tx}(v), \\ a_{xt}(-v) = -a_{tx}(v), \quad a_{xx}(-v) = a_{xx}(v). \end{gathered}$$

Recalling that S' is moving at v along the x-axis of S and the origin of S' coincides with the origin of S at $t = 0$. That is, when $x = vt$, then $x' = 0$, or

$$x' = \alpha(v)\,(x - vt)\,,$$

where

$$\alpha(v) = \alpha(-v) = a_{xx}(v).$$

Therefore

$$a_{xt} = -v\alpha(v).$$

Let's say

$$a_{tt}(v) = \gamma(v), \quad a_{tx}(v) = -\gamma(v)\zeta(v), \quad a_{yy}(v) = a_{zz}(v) = \phi(v).$$

We see

$$\gamma(v) = \gamma(-v), \quad -\gamma(v)\zeta(v) = \gamma(-v)\,\zeta(-v),$$
$$\zeta(v) = -\zeta(v), \quad \phi(v) = \phi(-v),$$

and

$$t' = \gamma(v)\,(t - \zeta(v)\,x)\,.$$

Now we can write the transformation in two 2-dimensional expressions.

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} \alpha(v) & -v\,\alpha(v) \\ -\gamma(v)\,\zeta(v) & \gamma(v) \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}, \quad \begin{pmatrix} y' \\ z' \end{pmatrix} = \begin{pmatrix} \phi(v) & 0 \\ 0 & \phi(v) \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} \tag{1.8}$$

Let's assume the reciprocity of the transformation. That means the transformation from S' to S moving at velocity $-v$ relative to S' will be

$$\begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} \alpha(-v) & v\,\alpha(-v) \\ -\gamma(-v)\,\zeta(-v) & \gamma(-v) \end{pmatrix} \begin{pmatrix} x' \\ t' \end{pmatrix}, \quad \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} \phi(-v) & 0 \\ 0 & \phi(-v) \end{pmatrix} \begin{pmatrix} y' \\ z' \end{pmatrix}$$

Recalling that $\alpha(v) = \alpha(-v)$, $\gamma(v) = \gamma(-v)$, $\zeta(v) = -\zeta(-v)$, $\phi(v) = \phi(-v)$, we find

$$\begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} \alpha(v) & v\,\alpha(v) \\ \gamma(v)\,\zeta(v) & \gamma(v) \end{pmatrix} \begin{pmatrix} x' \\ t' \end{pmatrix}, \quad \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} \phi(v) & 0 \\ 0 & \phi(v) \end{pmatrix} \begin{pmatrix} y' \\ z' \end{pmatrix}.$$

Plugging these into Eqs. (1.8), we find (omitting (v) for shorthand)

$$\begin{pmatrix} \alpha & -v\,\alpha \\ -\gamma\,\zeta & \gamma \end{pmatrix} \begin{pmatrix} \alpha & v\,\alpha \\ \gamma\,\zeta & \gamma \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \phi & 0 \\ 0 & \phi \end{pmatrix} \begin{pmatrix} \phi & 0 \\ 0 & \phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore

$$\alpha^2 - v\,\alpha\,\gamma\,\zeta = 1, \;\; v\,\alpha^2 - v\,\alpha\,\gamma = 0 \quad \rightarrow \quad \alpha = \gamma, \;\; \gamma^2\,(1 - v\,\zeta) = 1, \tag{1.9}$$
$$\phi^2 = 1 \quad \rightarrow \quad \phi = \pm 1. \tag{1.10}$$

The transformation should be the identity when $v = 0$, (Eq. (1.5)), we should choose $\phi = 1$. To summarize what we got so far,

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \gamma(v) \begin{pmatrix} 1 & -v \\ -\zeta(v) & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}, \quad \begin{pmatrix} y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}.$$

Now we require that two consecutive transformations yield the same class of transformation. That is

$$\gamma(u) \begin{pmatrix} 1 & -u \\ -\zeta(u) & 1 \end{pmatrix} \gamma(v) \begin{pmatrix} 1 & -v \\ -\zeta(v) & 1 \end{pmatrix} = \gamma(w) \begin{pmatrix} 1 & -w \\ -\zeta(w) & 1 \end{pmatrix}.$$

Proceeding with the calculation yields

$$\gamma(u)\,\gamma(v)\begin{pmatrix} 1+u\,\zeta(v) & -u-v \\ -\zeta(u)-\zeta(v) & 1+v\,\zeta(u) \end{pmatrix} = \gamma(w)\begin{pmatrix} 1 & -w \\ -\zeta(w) & 1 \end{pmatrix}. \tag{1.11}$$

From the diagonal elements,

$$\begin{aligned}
&\gamma(u)\,\gamma(v)\,(1+u\,\zeta(v)) = \gamma(w), \quad \gamma(u)\,\gamma(v)\,(1+v\,\zeta(u)) = \gamma(w),\\
\rightarrow \quad & 1+u\,\zeta(v) = 1+v\,\zeta(u) = \gamma(w)/\gamma(u)\gamma(v),\\
\rightarrow \quad & u\,\zeta(v) = v\,\zeta(u) \ \rightarrow\ \zeta(v)/v = \zeta(u)/u,
\end{aligned}$$

we see that $\zeta(v)/v$ is constant because $\zeta(v)/v = \zeta(u)/u$ for any v and u. Let's say $\zeta(v)/v = K$. From Eq. (1.9),

$$\gamma(v)^2\,(1-v\,\zeta(v)) = 1 \quad \rightarrow \quad \gamma(v) = \frac{1}{\sqrt{1-v^2K}}$$

Therefore, the transformation, Eq. (1.8), becomes

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \frac{1}{\sqrt{1-v^2K}}\begin{pmatrix} 1 & -v \\ -vK & 1 \end{pmatrix}\begin{pmatrix} x \\ t \end{pmatrix}, \quad \begin{pmatrix} y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix} \tag{1.12}$$

From Eq. (1.11),

$$\frac{1+u\,v\,K}{\sqrt{(1-u^2K)(1-v^2K)}}\begin{pmatrix} 1 & -\dfrac{u+v}{1+u\,v\,K} \\ -\dfrac{(u+v)K}{1+u\,v\,K} & 1 \end{pmatrix} = \frac{1}{\sqrt{1-w^2K}}\begin{pmatrix} 1 & -w \\ -wK & 1 \end{pmatrix}.$$

Now the inverse of the forefront factor of above is

$$\begin{aligned}
\frac{\sqrt{(1-u^2K)(1-v^2K)}}{1+u\,v\,K} &= \sqrt{\frac{1+u^2v^2K^2-(u^2+v^2)K}{1+u^2v^2K^2+2\,u\,v\,K}},\\
&= \sqrt{\frac{1+u^2v^2K^2+2\,u\,v\,K-(u^2+v^2+2\,u\,v)K}{1+u^2v^2K^2+2\,u\,v\,K}},\\
&= \sqrt{1-\frac{(u+v)^2K}{(1+u\,v\,K)^2}}.
\end{aligned}$$

Therefore

$$w = \frac{u+v}{1+u\,v\,K}. \tag{1.13}$$

In the case where $K=0$, above becomes Newtonian velocity addition law.

$$w = u+v$$

In the case where $K > 0$, Eq. (1.13) becomes the relativistic velocity addition law with $K = 1/c^2$.

$$w = \frac{u+v}{1+u\,v/c^2}$$

Finally

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \frac{1}{\sqrt{1-(v/c)^2}} \begin{pmatrix} 1 & -v \\ -v/c^2 & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}, \quad \begin{pmatrix} y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}$$

or using ct instead of t makes the matrix symmetric and the both element of the vector have the same dimension.

$$\begin{pmatrix} x' \\ ct' \end{pmatrix} = \frac{1}{\sqrt{1-(v/c)^2}} \begin{pmatrix} 1 & -v/c \\ -v/c & 1 \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix}, \quad \begin{pmatrix} y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}.$$

1.5 Mermin's derivation of velocity addition law

Mermin [3] demonstrated the way to observe the violation of classical velocity addition law, and derived the relativistic one without using the constancy of the speed of light. In this case, it does not have to be light; any object fast enough can demonstrate the violation of classical velocity addition law. Mermin's derivation also includes an arbitrary parameter K, which is also identified as $1/c^2$.

1.6 Vector notation

Let's consider an inertial frame K with coordinates $(ct, \boldsymbol{r})$, and another inertial frame K' with coordinates $(ct', \boldsymbol{r}')$. Suppose that K' is moving at velocity $\boldsymbol{V}$ relative to K and the origin of K' coincides with the origin of K at $t = 0$ and $t' = 0$. One trivial condition is that a point $\boldsymbol{r} = \boldsymbol{V}t$ is transformed to $\boldsymbol{r}' = 0$. From the homogeneity of space and time, we see that the transformation must be linear. From the isotropy of space, we see that vector components parallel to $\boldsymbol{V}$ are scaled differently than vector components normal to $\boldsymbol{V}$. Therefore the transformation should be written with real parameters η, α, γ, ζ,

$$\boldsymbol{r}' = \eta \left(1 - \frac{\boldsymbol{\beta}\,(\boldsymbol{\beta}\cdot\ )}{\beta^2}\right)\left(\boldsymbol{r} - \boldsymbol{\beta}ct\right) \;+\; \alpha\,\frac{\boldsymbol{\beta}\,(\boldsymbol{\beta}\cdot\ )}{\beta^2}\left(\boldsymbol{r} - \boldsymbol{\beta}\,ct\right), \quad ct' = \gamma\left(ct - \zeta\frac{(\boldsymbol{\beta}\cdot\ )}{\beta}\boldsymbol{r}\right),$$

where we used $\boldsymbol{\beta} = \boldsymbol{V}/c$ for shorthand. Note that

$$\frac{\boldsymbol{\beta}\,(\boldsymbol{\beta}\cdot\ )}{\beta^2}$$

is a projection operator which projects a vector onto the direction of $\boldsymbol{V}$.

$$\frac{\boldsymbol{\beta}\,(\boldsymbol{\beta}\cdot\ )}{\beta^2}\,\boldsymbol{r} = \frac{\boldsymbol{\beta}\,(\boldsymbol{\beta}\cdot\boldsymbol{r})}{\beta^2}$$

Proceeding with the calculation yields,

$$\boldsymbol{r}' = \eta \left(1 - \frac{\boldsymbol{\beta}\,(\boldsymbol{\beta}\cdot\ )}{\beta^2}\right)\boldsymbol{r} \;+\; \alpha\left(\frac{\boldsymbol{\beta}\,(\boldsymbol{\beta}\cdot\boldsymbol{r})}{\beta^2} - \boldsymbol{\beta}\,ct\right), \quad ct' = \gamma\left(ct - \zeta\frac{(\boldsymbol{\beta}\cdot\boldsymbol{r})}{\beta}\right),$$

or

$$\boldsymbol{r}' = \eta\,\boldsymbol{r}_\perp \;+\; \alpha\left(\boldsymbol{r}_\parallel - \boldsymbol{\beta}\,ct\right), \quad ct' = \gamma\left(ct - \zeta\frac{(\boldsymbol{\beta}\cdot\boldsymbol{r})}{\beta}\right),$$

where

$$\boldsymbol{r}_\perp = \left(1 - \frac{\boldsymbol{\beta}\,(\boldsymbol{\beta}\cdot\ )}{\beta^2}\right)\boldsymbol{r}, \quad \boldsymbol{r}_\parallel = \frac{\boldsymbol{\beta}\,(\boldsymbol{\beta}\cdot\boldsymbol{r})}{\beta^2}.$$

Now we invoke the condition that interval is invariant, $(ct')^2 - (r')^2 = (ct)^2 - r^2$.

$$\begin{aligned}
(ct')^2 &= \gamma^2\left((ct)^2 + \zeta^2\frac{(\boldsymbol{\beta}\cdot\boldsymbol{r})^2}{\beta^2} - 2\,ct\,\zeta\,\frac{(\boldsymbol{\beta}\cdot\boldsymbol{r})}{\beta}\right),\\
&= \gamma^2(ct)^2 + \gamma^2\zeta^2\frac{(\boldsymbol{\beta}\cdot\boldsymbol{r})^2}{\beta^2} - 2\,ct\,\gamma^2\,\zeta\,\frac{(\boldsymbol{\beta}\cdot\boldsymbol{r})}{\beta},\\
(r')^2 &= \eta^2 r_\perp^2 + \alpha^2\left(\boldsymbol{r}_\parallel - \boldsymbol{\beta}\,ct\right)^2,\\
&= \eta^2\left(\boldsymbol{r} - \frac{\boldsymbol{\beta}\,(\boldsymbol{\beta}\cdot\boldsymbol{r})}{\beta^2}\right)^2 + \alpha^2 r_\parallel^2 + \alpha^2\beta^2(ct)^2 - 2\,ct\,\alpha^2\,(\boldsymbol{\beta}\cdot\boldsymbol{r}_\parallel),\\
&= \alpha^2\beta^2(ct)^2 + \eta^2\,r^2 + \left(\alpha^2 - \eta^2\right)\frac{(\boldsymbol{\beta}\cdot\boldsymbol{r})^2}{\beta^2} - 2\,ct\,\alpha^2\beta\,\frac{(\boldsymbol{\beta}\cdot\boldsymbol{r})}{\beta}
\end{aligned}$$

To make $(ct')^2 - (r')^2$ equal to $(ct)^2 - r^2$, we want

$$\gamma^2 - \alpha^2\beta^2 = 1, \quad \eta^2 = 1, \quad \gamma^2\zeta^2 - \alpha^2 + \eta^2 = 0, \quad \gamma^2\zeta - \alpha^2\beta = 0. \tag{1.14}$$

From the last equation, we find

$$\alpha^2 = \frac{\gamma^2\zeta}{\beta}.$$

Inserting this and $\eta^2 = 1$ into the first and the third equations of Eqs. (1.14), we find

$$\zeta\beta = 1 - \frac{1}{\gamma^2}, \quad \gamma^2\zeta\beta\,(1 - \zeta\beta) = \beta^2.$$

Inserting the first equation to the second, we find

$$\gamma^2 = \frac{1}{1-\beta^2}, \quad \zeta = \beta, \quad \alpha^2 = \gamma^2, \quad \eta^2 = 1.$$

Since the transformation has to be the identity when $\boldsymbol{V} = 0$, we choose positive values.

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}, \quad \alpha = \gamma, \quad \eta = 1.$$

Finally the transformation is

$$\boldsymbol{r}' = \boldsymbol{r}_\perp \ + \ \gamma \left(\boldsymbol{r}_\parallel - \boldsymbol{\beta}\, ct\right), \quad ct' = \gamma \left(ct - (\boldsymbol{\beta} \cdot \boldsymbol{r})\right),$$

where

$$\boldsymbol{r}_\perp = \left(\boldsymbol{r} - \boldsymbol{r}_\parallel\right), \quad \boldsymbol{r}_\parallel = \frac{\boldsymbol{\beta}\,(\boldsymbol{\beta} \cdot \boldsymbol{r})}{\beta^2}, \quad \boldsymbol{\beta} = \boldsymbol{V}/c, \quad \gamma = \frac{1}{\sqrt{1 - \beta^2}}.$$

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